ON THE MAPPING OF LOCALLY CONNECTED CONTINUA INTO SIMPLE ARCS

BEATRICE AITCHISON

A dissertation submitted to the Board of University Studies, the Johns Hopkins University, in conformity with the requirements for the degree of Doctor of Philosophy

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O przekształcaniu continuów lokalnie spójnych na łuki proste.

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Przedstawił K. Kuratowski dnia 1 grudnia 1934 r.

Streszczenie.

Przedmiotem badań tej pracy jest struktura continuów lokalnie spójnych, które dają się przekształcić w odcinek z pomocą funkcji ciągłej, przyjmującej każdą swą wartość skończoną ilość razy.

Beatrice Aitchison (Baltimore, U.S.A.).

On the mapping of locally connected continua into simple arcs.

Mémoire présenté par M. C. Kuratowski dans la séance du 1 décembre 1934.

1. Čech has proposed the problem: to find necessary and sufficient conditions for a real continuous function $f[x]$ to be defined over a locally connected continuum M, such that $f[x]=c$ shall have only a finite number of solutions in M. He states three necessary conditions: that M must be a regular curve in the sense of Menger, that the set of end points of M must be countable, and that, if R is the set of points of M of order greater than 2, R is not dense on any subcontinuum of M[1]). Mazurkiewicz assumed that the space M under consideration was a dendrite, and has found a necessary and sufficient condition for such a space to be transformed continuously into an arc, so that only a finite number of points of M go into the same point of the arc: the set of end points of M must be countable and, if Φ-derivation of R, the set of branch points of M, yields that subset of R, each point of which is a bilateral limit point of R, some Φ-derivative of R must vanish [2]). The notion of Φ-derivation will be defined in detail later.

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[1]) „Une classe de continus", *Fundamenta Mathematicae*, Vol. 18 (1932), p. 85.

[2]) „Une classe de dendrites", Fund. Math., Vol. 18 (1932), p. 88.

2. A property Q is said to be cyclicly extendible if, when every cyclic element of a space has property Q, the whole space has the property [1]). The purpose of this paper is to show certain sufficient conditions for the cyclic extendibility of the property (property P) that a real continuous function may be defined over a continuum, assuming any value only a finite number of times.

Definitions. The cyclic elements of a locally connected continuum M are: a) the cut points of M, and b) the sets of points S_p, where p is any non-cut point of M and S_p is the set of points of M, no one of which can be separated from p by any single point of M. The cyclic chain $C[a, b]$ between two points a and b of M is equal to the sum of the two points, the set K of all points separating them in M, and all cyclic elements C, such that C contains exactly two points of $K+a+b$. There are at most a countable number of such elements C. A continuum M is said to have property P, if there exists a real continuous function $f[x]$, defined over M, such that $f[x]=c$ (where c is a real number) has only a finite number of solutions in M. We shall call such a function $f[x]$ a real continuous function of finite sections.

3. **Lemma** 3.1. *If there is defined over a continuum C a real continuous function $f[x]$ such that $f[x]=c$ has only a finite number of solutions in C, then, if $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_m]$ are two distinct sets of points in C, such that $f[a_i] \neq f[b_j]$ $[i=1, \ldots, n;\ j=i, \ldots, m]$, there exists a real continuous function $g[x]$ of finite sections defined over C, such that if t is the simple arc which is the transform of C under $g[x]$ then every a_i corresponds to one end point of t and every b_j to the other end point.*

Proof. Suppose that $f[C]=xy$, $f[a_i]=a_i'$, and $f[b_j]=b_j'$ $[i=1, \ldots, n;\ j=1, \ldots, m]$, where the a s are so ordered that $x \leqq a_1' \leqq a_2' \leqq \cdots \leqq a_n' \leqq y$. If a_1' is not an end point of xy, there exists a homeomorphism t_{1a_1}, transforming the arc xa_1' into the arc $a_1'y$, so that x goes into y and a_1' into a_1'. Let $t_{a_1}[w]=t_{1a_1}[w]$, $w \in xa_1'$ and $t_{a_1}[w]=w$, $w \in a_1'y$. There exists

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[1]) The most complete reference on the subject of cyclic éléments is to be found in the paper of Kuratowski and Whyburn, „Sur les elements cycliques...", Fund. Math., Vol. 16 (1930), pp. 305—331.

a point z_1 of $<a_1' a_2'>$ (where $<a_1' a_2'>$ denotes the set $a_1' a_2' - a_1' - a_2'$), a homeomorphism t_{1a_2} such that $t_{1a_2}[a_1' z_1] = z_1 a_2'$, where $t_{1a_1}[a_1'] = a_2'$ and $t_{1a_1}[z_1] = z_1$, and a homeomorphism t_{2a_2} such that $t_{2a_2}[z_1 a_2'] = a_2' y$, where $t_{2a_2}[z_1] = y$ and $t_{2a_2}[a_2'] = a_2'$. Then if $t_{a_2}[w] = t_{1a_2} t_{2a_2}[w]$, $w \in a_1' a_2'$, and $t_{a_2}[w] = w$, $w \in a_2' y$, $t_{a_2}[a_1' y] = a_2' y$ and is a continuous transformation of finite sections. Moreover, $t_{a_2}[a_1'] = a_2'$ and a_1' and a_2' have been sent into the same end point of the arc.

Repeat this process n times. If $a_n' = y$, then a_{n-1}' is the other end point of the arc after $n-1$ or fewer transformations. We can choose a point d interior to the arc $a_{n-1}' a_n'$. There exists a homeomorphism t_{1a_n} such that $t_{1a_n}[d a_n'] = d a_{n-1}'$, and a_n' goes into a_{n-1}'. Then $t_{a_n}[w] = t_{1a_n}[w]$, $w \in d a_n'$ and $t_{a_n}[w] = w$, $w \in d a_{n-1}'$ is a continuous function of finite sections.

Now no b_j' can be an a_i', so that after we have $[a_1', a_2', \ldots, a_n']$ in one end point of the arc, that end point will not be disturbed when we operate on the resultant b_j''s in exactly the same manner, after relabelling them so that $a_1' \leqq b_1'' \leqq \cdots \leqq b_m'' \leqq y'$. After at most m transformations, every b_j'' will have been sent into the other end point of the arc.

If $g[x] = f t_{a_1} t_{a_2} \ldots t_{a_n} t_{b_1} \ldots t_{b_m}[x]$, then $g[x]$ is obviously real and continuous and of finite sections.

Lemma 3.2. *Let C be a continuum having property P. If x and y are two points of C, there exists a continuous transformation of finite sections of C into an arc $x'y'$, such that x is the only point of C transformed into x' and y is the only point transformed into y'.*

Proof. From Čech's results we know that x and y are regular points of C. There exists an $\varepsilon_1 > 0$, such that suitably chosen ε_1-regions of x and y have no points in common and have finite boundaries. If the images of these two boundaries under $f[z]$ (where $f[z]$ is the transformation implied under the hypothesis that C has property P) are two distinct sets of points, let $\varepsilon_1' = \varepsilon_1$. If, however, the images of the boundaries of $R_{\varepsilon_1}[x]$ and $R_{\varepsilon_1}[y]$ do not go into distinct sets, there can be at most a finite number of regions of y with boundary points such that under $f[z]$ one or more goes into a point which is the

image of a boundary point of the ε_1-region of x chosen. Therefore we can find an $\varepsilon_1' < \varepsilon_1$, such that $f[F[R_{\varepsilon_1}[x]]] \cdot f[F[R_{\varepsilon_1'}[y]]] = 0$. In the following statements we shall assume that for each ε_i, we have chosen an $\varepsilon_{i+1} < \frac{\varepsilon_i}{2}$, such that there exists an ε_{i+1}-region $R_{\varepsilon_{i+1}}[x]$ [or $R_{\varepsilon_{i+1}}[y]$] so that $f[F[R_{\varepsilon_i}[x]]] \cdot f[F[R_{\varepsilon_{i+1}}[x]]] = 0$.

If $G_1 = C - R_{\varepsilon_1}[x] - R_{\varepsilon_1'}[y]$, then G_1 has at most a finite number of components. Let D_1 be one of these. Then D_1 can be sent by $f[z]$ into a subarc of ab, where $f[C] = ab$, so that all those points of D_1 which are boundary points of $R_{\varepsilon_1}[x]$ go into points distinct from those which are the images of boundary points of $R_{\varepsilon_1'}[y]$. By lemma 3.1 we can transform the arc into the interval $[0, 1]$, so that all points which are images of $F[R_{\varepsilon_1'}[y]]$ go into 1, and all those which are images of $F[R_{\varepsilon_1}[x]]$ go into 0. We can do likewise with the other components of G_1, transforming the boundary points into the proper end points and then transforming all of the arcs so obtained into $[0, 1]$. The whole transformation $g_1[z]$, defined for $z \in G_1$, will obviously be continuous and of finite sections.

Take $\varepsilon_2 < \frac{\varepsilon_1}{2}$, and so that $R_{\varepsilon_2}[x]$ and $R_{\varepsilon_2}[y]$ have finite boundaries, and $\overline{R_{\varepsilon_2}[y]} \subset R_{\varepsilon_1}[y]$ and $\overline{R_{\varepsilon_2}[x]} \subset R_{\varepsilon_1}[x]$. Then $G_2 = \overline{R_{\varepsilon_1}[x]} - R_{\varepsilon_2}[x]$ has at most a finite number of components, which can be sent by $f[z]$ and a continuous transformation of finite sections into the interval $[-1/2, 0]$, so that all points on the boundary of $R_{\varepsilon_2}[x]$ go into $-1/2$ and all points on the boundary of $R_{\varepsilon_1}[x]$ go into 0. The whole transformation $g_2[z]$ defined over G_2, is continuous and of finite sections. Similarly we can obtain by a transformation $h_2[z]$ of $H_2 = \overline{R_{\varepsilon_1'}[y]} - R_{\varepsilon_2}[y]$ the interval $[1, 3/2]$, such that 1 is the image of the boundary of $R_{\varepsilon_1'}[y]$ and $3/2$ is the image of the boundary of $R_{\varepsilon_2}[y]$.

In general, having obtained by the transformations $g_1[z]$, $\ldots$, $g_i[z]$, $h_2[z]$, $\ldots$, $h_i[z]$ the interval $\left[\frac{1-2^{i-1}}{2^{i-1}}, \frac{2^i-1}{2^{i-1}}\right]$ which is the image of $C - R_{\varepsilon_i}[x] - R_{\varepsilon_i}[y]$, we take $\varepsilon_{i+1} < \frac{\varepsilon_i}{2}$, so that

the boundaries of $R_{\varepsilon_{i+1}}[x]$ and $R_{\varepsilon_{i+1}}[y]$ are finite and $\overline{R_{\varepsilon_{i+1}}[x]} \subset R_{\varepsilon_i}[x]$ and $\overline{R_{\varepsilon_{i+1}}[y]} \subset R_{\varepsilon_i}[y]$.

Then $G_{i+1} = \overline{R_{\varepsilon_i}[x]} - R_{\varepsilon_{i+1}}[x]$ has at most a finite number of components, which can be transformed by $f[z]$ and a continuous transformation of finite sections into the interval $\left[\frac{1-2^i}{2^i}, \frac{1-2^{i-1}}{2^{i-1}}\right]$, so that all points on the boundary of $R_{\varepsilon_i}[x]$ go into $\frac{1-2^{i-1}}{2^{i-1}}$ and all the points on the boundary of $R_{\varepsilon_{i+1}}[x]$ go into $\frac{1-2^i}{2^i}$. The whole transformation will be $g_{i+1}[z]$, which is defined over G_{i+1}. A similar interval $\left[\frac{2^i-1}{2^{i-1}}, \frac{2^{i+1}-1}{2^i}\right]$ can be obtained by a transformation $h_{i+1}[z]$, which will be the image of $H_{i+1} = \overline{R_{\varepsilon_i}[y]} - R_{\varepsilon_{i+1}}[y]$.

Continuing indefinitely, we see that $\lim_{i\to\infty}\left[\frac{1-2^i}{2^i}, \frac{1-2^{i-1}}{2^{i-1}}\right] = -1$, which we shall define as the image of x. Then -1 is the image of x alone, for if it were the image of another point $z \neq y$ of C, for some i, $z \in G_i$ or H_i, and $g_i[z] \subset \left[\frac{1-2^{i-1}}{2^{i-1}}, \frac{1-2^{i-2}}{2^{i-2}}\right]$ or $h_i[z] \subset \left[\frac{2^{i-1}-1}{2^{i-2}}, \frac{2^i-1}{2^{i-1}}\right]$. But z can have but one image which thus cannot be -1. Similarly $\lim_{i\to\infty}\left[\frac{2^{i-1}-1}{2^{i-2}}, \frac{2^i-1}{2^{i-1}}\right] = 2$, which is defined as the image of y and is the image of y alone.

Then $g[z]$ is defined in the following manner: if $z \neq x$ or y, and i is the smallest integer such that $z \in G_i$ or H_i, let $g[z] = g_i[z]$ or $h_i[z]$. If $z = x$, $g[z] = -1$ and if $z = y$, $g[z] = 2$.

It is easily seen that $g[z]$ is continuous over C. Also $g[z]$ is of finite sections. The points $\frac{1-2^{i-1}}{2^{i-1}}$ and $\frac{2^i-1}{2^{i-1}}$ are the only ones about which there might be doubt in this respect. But $\frac{1-2^{i-1}}{2^{i-1}}$ or $\frac{2^i-1}{2^{i-1}}$ belongs to just two intervals and is the image of only a finite number of points from each of the sets defining the intervals.

We have now proved that if a continuum C has property P, it can be transformed into a simple arc by a continuous transformation of finite sections, so that any two given points of C will go into the end points of the arc, and no other point of C will be transformed into these points.

4. **Lemma** 4.1. *If every cyclic element of a cyclic chain $C[a, b]$ between two points a and b in a locally connected continuum has property P, then $C[a, b]$ can be transformed continuously by a function $f[x]$ of finite sections into an arc contained in the chain, such that $f^{-1}[f[x]] = x$ when $x \in K + a + b$, and $C[a, b]$ has property P.*

Proof. Let $C[a, b] = K + a + b + \sum_i C_i$, where C_i contains exactly two points x_i and y_i of $K + a + b$. By hypothesis there may be defined over each nondegenerate cyclic element C_i of $C[a, b]$ a real continuous function of finite sections. By lemma 3.2 there is a continuous transformation of finite sections, $f_i[z]$ of C_i into an arc $x_i y_i$ of itself, such that x_i is the only point of C_i, going into x_i and y_i is the only point going into y_i. Let the transformation $f[x]$ be defined as follows: $f[x] = x$, $x \in K + a + b$ and $f[x] = f_i[x]$, $x \in C_i$. Clearly $f[x]$ is of finite sections. It is also continuous. Suppose that $\lim_{i \to \infty} [z_i] = z$. Two cases are possible: a) the point z may be a point of some cyclic element C_k but not a point of $K + a + b$. Then $[z_i] \subset C_k$ for $i >$ some N, and $\lim_{i \to \infty} [f[z_i]] = \lim_{i \to \infty} [f_k[z_i]] = f_k[z] = f[z]$; b) the point z may be a point of $K + a + b$. If infinitely many points of $[z_i]$ are contained in a cyclic element C_k containing z, then as before $\lim_{i \to \infty} [f[z_i]] = f[z]$. If $[z_i] \subset K + a + b$, then $\lim_{i \to \infty} [f[z_i]] = \lim_{i \to \infty} [z_i] = z = f[z]$. If $[z_i]$ is contained in infinitely many different cyclic elements C_k, as $\lim_{k \to \infty} \delta[C_k] = 0$, z must be the only limit point of $[C_k]$. Then as $f[z_i] \subset C_i$ (it may be assumed that $z_i \subset C_i$), $\lim_{i \to \infty} [f[z_i]] = z = f[z]$.

Then $f[C[a, b]] = K + a + b + \sum_i x_i y_i = t$, where the arcs $x_i y_i$ have only their end points in common with $K + a + b$. As t is the continuous image of a closed, compact, connected and locally connected set, it is itself closed, compact, connected

and locally connected and is, moreover, an arc in $C[a, b]$. Then t can be transformed by a homeomorphism $T[x]$ into the unit interval [0, 1]. The function $f \cdot T[x]$ is real, continuous and of finite sections over $C[a, b]$ and $C[a, b]$ has property P.

G. T. Whyburn has shown [1]) that in a locally connected continuum S there exist two distinct sequences of points $[p_i]$ and $[q_i]$, such that the cyclic chains $C[p_i, q_i]$ have the following properties:

a) Each p_i is contained in a cyclic element which is not interior to any cyclic chain of S.

b) $C[p_m, q_m] \cdot \sum_{i=1}^{m-1} C[p_i, q_i] = q_m$.

c) If $H_n = \sum_{i=1}^{n} C[p_i, q_i]$ and $H = \sum_{n=1}^{\infty} H_n = \sum_{i=1}^{\infty} C[p_i, q_i]$, then every point of $S - H$ is an end point.

d) The diameters of the components of $S - H_n$ go to zero with $1/n$.

e) $\lim_{i \to \infty} \delta [C[p_i, q_i]] = 0$.

If S has but a countable number of end points, then this decomposition may be made so that $S = H = \overline{H}$.

5. Definitions. The set X of branch elements of a space H is the set of all non-degenerate cyclic elements Z, such that $H - Z$ contains at least three components, the degenerate cyclic elements Z, lying on no nondegenerate cyclic element, such that the complement of Z contains at least three components, and the degenerate cyclic elements Z, such that Z is the only cut point of H on some nondegenerate cyclic element and $H - Z$ contains exactly three components. A bilateral limit element x of elements of X of H is one such that for two components H_1 and H_2 of $H - X$ there exist two points a and b (not necessarily distinct) of x, such that $a \subset \limsup (H_1 \cdot X)$ and $b \subset \limsup (H_2 \cdot X)$. Bilateral limit points of a subset U of a dendrite T are all points b of T such that for two components T_1 and T_2 of $T - b$, $b \subset \limsup (T_1 \cdot U) \cdot \limsup (T_2 \cdot U)$.

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[1]) ibid., p. 319, and for property a) see Whyburn, „Concerning the structure of continuous curves", *Amer. Jour. Math.*, Vol. L (1928), pp. 167—194, especially theorems 13 and 14.

We shall now define for a dendrite D the notion of Φ-reducibility, introduced by Mazurkiewicz. Let U be a subset of D. Let $\Phi(U)$ be the bilateral coherence of U; i. e., all points of U which are bilateral limit points of U. Then $\Phi_0[U] = U$ $\Phi_\alpha[U] = \Phi[\Phi_{\alpha-1}[U]]$, where α is an ordinal number of the first species, and $\Phi_\alpha[U] = \prod_{\beta<\alpha} \Phi_\beta[U]$, where α is an ordinal number of the second species. Moreover, there is an ordinal number $\lambda[U] < +\Omega$, such that $\Phi_{\lambda(u)}[U] = \Phi_{\lambda(u)+1}[U]$. This is evident when U is countable, as it will be in our discussion. If $\Phi_{\lambda(u)}[U] = 0$, U is said to be reducible by the operation Φ, or Φ-reducible.

An extension of this operation may be made to the branch elements of a locally connected continuum. If X is the set of all branch elements of a space H, let $\Phi'[X]$ be the bilateral coherence of X; i. e., the set of all bilateral limit elements of X, belonging to X. Then $\Phi_0'[X] = X$, $\Phi_\alpha'[X] = \Phi'[\Phi_{\alpha\ 1}'[X]]$, where α is an ordinal number of the first species, and $\Phi_\alpha'[X] = \prod_{\beta<\alpha} \Phi_\beta'[X]$, where α is an ordinal number of the second species. Moreover, there is an ordinal number $\lambda[X] < +\Omega$, such that $\Phi_{\lambda(x)}'[X] = \Phi_{\lambda(x)+1}'[X]$. This is evident when X is countable, the only case to be considered here. If $\Phi_{\lambda(x)}'(X) = 0$, X is said to be Φ'-reducib'e.

If our space H is a dendrite, it is evident that the Φ' operation is identical with the Φ operation on the branch points of H.

Suppose a) that H is a locally connected continuum containing only a countable number of end points; b) that there is defined over each cyclic element C of H a real continuous function of finite sections, such that the set of cut points of H on C is transformed into a set reducible in the ordinary sense; and c) that the set of branch elements of H is Φ'-reducible. Under these conditions there can be defined over H a real continuous function of finite sections (or H can be sent into a simple arc by a continuous transformation of finite sections).

The proof will consist of two parts: a) a definition of a transformation of H into a dendrite D by means of a continuous transformation of finite sections, and b) a demonstration that this dendrite has property P, in which case H also has property P.

Proof. By hypothesis, $H=\sum_{i=1}^{\infty} C_i$, where $C_i=C[p_i, q_i]$. By lemmas 3.1, 3.2, and 4.1, there exists a continuous transformation $f_1[x]$ of finite sections sending C_1 into an interval $p_1'q_1'$ of the plane π, of length <1, so that $f_1[p_1]=p_1'$, $f_1[q_1]=q_1'$, $f_1^{-1}[p_1']=p_1$, and $f_1^{-1}[q_1']=q_1$, and so that the set of cut points of H on any cyclic element of C_1 goes into a reducible set on the subarc of $p_1'q_1'$ corresponding to the element. Then $p_1'q_1'$ is contained in a connected open set R_1 of π, with boundary a simple closed curve J_1, every point of which is at a distance <1 from C_1'.

Then $q_2'\subset p_1'q_1'$. There exists a continuous transformation $f_2[x]$ of finite sections sending $C[p_2, q_2]$ into an interval $p_2'q_2'=C_2'$ contained wholly in R_1, of length $<1/2$, and $<\delta[C_1']$, such that $C_2'\cdot C_1''=q_2'$. There exists a simple closed curve $J_2\subset R_1$, separating p_2' and p_1', and enclosing $C_2'-q_2'$, such that every point of J_2 is at a distance $<1/2$ from C_2', with a connected open interior R_2. Then $\bar{R}_2\cdot C_1'=q_2'$.

In general, $q_n'\subset\sum_{j=1}^{n-1} p_j'q_j'$. If i is the greatest integer such that $q_n'\subset R_i$, then there exists a continuous transformation $f_n[x]$ of finite sections sending $C[p_n, q_n]$ into an interval $p_n'q_n'$ contained wholly in R_i, of length $<\frac{1}{2^{n-1}}$, and $<\delta[C_j']$ $[j=1, \ldots, n-1]$, such that $p_n'q_n'\cdot\sum_{j=1}^{n-1} p_j'q_j'=q_n'$ and so that the set of cut points of H on any cyclic element of C_n goes into a reducible set on $p_n'q_n'$. Then in $R_i-R_i\cdot\sum_{j=i+1}^{n-1} R_j$, we can separate p_n' and q_i' by a simple closed curve J_n, enclosing $C_n'-q_n'$, every point of which is at a distance $<\frac{1}{2^{n-1}}$ from C_n', and $J_n\cdot\sum_{j=1}^{n} C_j'=q_n'$. Then J_n is the boundary of a connected open region R_n containing $p_n'q_n'=q_n'$ and $\bar{R}_n\cdot\sum_{j=1}^{n-1} C_j'=q_n$.

Let $\sum_{i=1}^{\infty} p_i' q_i' = D$, where i may or may not become infinite. The transformation sending H into D is defined as follows: if $x \in C_n \subset H$, $f[x] = f_n[x]$. The function is of finite sections. If x is not a q_i, it is contained in C_n alone, and $f_n^{-1}[x']$ is finite. If x is a q_i, then q_i' is the image of but a finite number of points from the first $i-1$ transformations, of one point in the $i-$th transformation, and of no points or of just q_i again in the transformations after the $i-$th. The transformation is also continuous. Let $\lim_{i \to \infty} [x_i] = x$ in H. Let k be the first integer such that $x \subset C_k$. Then $x' \subset C_k'$. If all $[x_i] \subset C_k$, then $[x_i'] \subset C_k'$ and $\lim_{i \to \infty} [x_i'] = x'$, as $f_k(z)$ is continuous. If all $[x_i]$ are not contained in C_k, but an infinite number are contained in C_j, then $j > k$, and $x \subset C_j$ and $\lim_{i \to \infty} [x_i'] = x'$, as $f_j[z]$ is continuous. If there are at most a finite number of $[x_i]$ in any one chain, we may suppose without loss of generality that there is just one x_i in a chain. Let $x_i \subset C_{n_i}$. Then $\lim_{i \to \infty} \delta [C_{n_i}] = 0$. If $x \subset C_{n_i}$ for an infinite number of $i's$, there exists an $\varepsilon > 0$ and a j, such that, for $i > j$, $\rho [x_i' x'] < \varepsilon$. If x is not contained in an infinite number of the $C_{n_i}'s$, we may suppose without loss of generality that x is not contained in any C_{n_i}. Let us take the arc $x_i' x$ contained in the dendrite $D_{n_i} = \sum_{j=1}^{n_i} C_j' = \sum_{j=1}^{n_i} p_j' q_j'$, and let y_i' be the first point it has in common with a C_{m_i}' containing x'. Then we obtain a sequence of points $[y_i']$, such that $\rho [y_i' x'] \to 0$ with $1/i$. Moreover, by the method of construction of D, and as one inverse of y_i' is a q, the points of any interval added on to C_{m_i}' at y_i' must be at a distance $< \frac{1}{2^{m_i - 1}}$ from C_{m_i}'. Hence any point of any interval joined to C_{m_i}', even through another interval, at y_i' must be at a distance $< \frac{2}{2^{m_i - 1}} = \frac{1}{2^{m_i - 2}}$ from y_i'. Hence $\rho [x_i' y_i'] < \frac{1}{2^{m_i - 2}}$. Therefore $\lim_{i \to \infty} \rho [x_i' y_i'] = 0$. But as $\rho [x_i' x'] < \rho [x_i' y_i'] + \rho [y_i' x']$, we see that $\lim_{i \to \infty} \rho [x_i' x'] = 0$, and that $f[x]$ is continuous over H.

The set D is self compact, connected and locally connected, as it is the image of a self compact, connected and locally connected set under a continuous transformation.

The set D contains no simple closed curve. Let $a+b \subset D$. Then $a+b \subset D_k$ for some $k[D_k = f[H_k]]$, and $a+b \subset ab$, an arc in D_k. If there were another arc t with a and b as endpoints, it would be contained partly in $D-D_k$. But any component of $D-D_k$ has just one limit point in D_k. If t contains a point x of $D-D_k$, then x is contained in some component C of $D-D_k$, and if $F[C]=y$, $ax \cdot xb \supset x+y$ and not just x.

Thus we see that by a continuous transformation of finite sections we can send H into a dendrite D. This dendrite, it is seen at once, fulfills one of Mazurkiewicz's sufficient conditions; i. e., has only a countable number of end points. We shall proceed to show that it fulfills also the second condition, that the set of branch points be reducible by the operation of bilateral coherence.

Lemma 5.1. *The transformation $f[x]$ sending H into D yields a homeomorphism $g[Z]$ between the elements Z of X and the corresponding elements Z' of $f[X]=X'$ (where the whole element Z and its transform Z' are considered as „points").*

Proof. If Z is a nondegenerate element of X, by the method of construction of D there is one and only one interval Z' of D corresponding to it, and Z' is not the image of any other nondegenerate cyclic element of H. If Z is a degenerate cyclic element of X, then also by the method of construction, it corresponds to just one point Z' of D which is not the image of any other point of H.

We know that $f[x]$ is pointwise continuous from X to X'. Hence $g[Z]$ is one to one and continuous from X to X'. It may be easily proved that $g[Z']$ is continuous also, thus proving that $g[Z]$ is a homeomorphism between these elements.

If the set X' is called the set of branch intervals of D, then no other interval of $\hat{D}-X'$, where $\hat{D}$ is the set of intervals corresponding to the cyclic elements of H, corresponding to a cyclic element of H will have more than two components in its complement. For if there were such an interval which did, it would contain a branch point y' of D. Then y' is the image of

some cut point y of H. If y' is not on a nondegenerate cyclic element of H, y' would be an element of X'. If y is on a nondegenerate cyclic element C of H, either C would be a branch element, and $I=f[C]$ would be an element of X' or y is the only cut point of H on C, and has exactly three components in its complement, thus being an element of X. Thus no interval of D in $\widehat{D}-X'$ contains any branch point of D.

Lemma 5.2. *If X is Φ'-reducible, then X' is Φ'-reducible.*

Proof. As an immediate consequence of lemma 5.1, no sequence of elements of X' can be convergent unless the corresponding sequence of elements of X was, and vice versa. Hence if X is Φ'-reducible, X' is of necessity Φ'-reducible, as the Φ' operation depends on this property. Thus if $\Phi'_\alpha[X]=0$, then $\Phi'_\alpha[X']=0$, as, in D, X' contains all of the bilateral limit intervals corresponding to bilateral limit elements of H and no more.

Now let R denote the set of all branch points of D. Then R is contained in the point set of X'. Suppose x', $x'\varepsilon R$, is the transform of some cut points of H (at most a finite number, all on one cyclic element) which are on a countable (finite or otherwise) number of cyclic elements of H. Then if I_i denotes the transform of C_i, one of such cyclic elements, $x'\subset I_i$, $i=1, 2, \ldots$, and there exists an ordinal number $\alpha_i<+\Omega$, such that $[I_i\cdot R]^{\alpha_i}=0$. Let $I_{x'}=\sum_i I_i$. If x' is on n intervals I_i only, then $[R\cdot I_{x'}]^{\alpha_1+\cdots+\alpha_n}=0$. If n is infinite, suppose that for $\alpha_{x'}$, $\alpha_i<\alpha_{x'}<+\Omega$ for all i, $[R\cdot I_{x'}]^{\alpha_{x'}}\cdot x'\neq 0$. As $[R\cdot I_i]^{\alpha_{x'}}\cdot x'=0$, and as $\lim_{i\to\infty}\delta[I_i]=0$, then x' is the only point which can be a limit point of all the derivatives of $R\cdot I_{x'}$ before the $\alpha_{x'}-th$ and hence $[R\cdot I_{x'}]^{\alpha_{x'}+1}\cdot x'=0$. Moreover, $\Phi_{\alpha_{x'}+1}[R\cdot I_{x'}]\cdot x'=0$. Let $\alpha_{x'}+1=\gamma_{x'}$.

However, if $\Phi_{\gamma_{x'}}[R]\cdot x'=x'$, there must be two components D_1 and D_2 of $D-I_{x'}$, such that $x'\subset \limsup D_1\cdot X'$ and $x'\subset \limsup D_2\cdot X'$.

Perform this derivation on every branch point x' of D. As R is countable, there exists an ordinal number α, such that $\gamma_{x'}<\alpha<+\Omega$, for all x'. Then $\Phi_\alpha[I_{x'}]\cdot x'=0$ for all x'.

Let the elements of X' be ordered into a sequence, Z_1', Z_2', ... For any element Z_i' there exists an ordinal number $\beta_i < +\Omega$, such that $\Phi'_{\beta_i}[X'] \cdot Z_i' = 0$. If x' is any element of R contained in Z_i', then $x' \cdot \Phi_{\alpha+\beta_i}[R] = 0$, because for some $\varepsilon > 0$, there will be intervals of $\Phi'_\gamma[X']$, for all $\gamma < \beta_i$, in at most one component of the complement of x' in its ε-neighborhood. Hence there will be points of $\Phi_{\alpha+\gamma}[R]$, for all $\gamma < \beta_i$, in at most one component of the complement of x' in its ε-neighborhood having x' as a limit point, for x' is not even an ordinary limit point of $\Phi_\alpha[I_{x'} \cdot R]$, and there can be no points of $\Phi_{\alpha+\gamma}[R]$, $\gamma < \beta_i$, on this type of interval, having x' as a limit point. Thus x' is not a bilateral limit point of $\Phi_{\alpha+\gamma}[R]$ for all $\gamma < \beta_i$.

As there are only a countable number of elements in X', there exists an ordinal number $\beta < +\Omega$, greater than all β_i, such that $\Phi'_\beta[X'] \cdot Z_i' = 0$ for all Z_i'. Then $\Phi_{\alpha+\beta}[R] \cdot x' = 0$ for all x', as each branch point of D is contained in an element of X'. Thus $\Phi_{\alpha+\beta}[R] = 0$, and R is Φ-reducible.

We have now shown that D fulfills Mazurkiewicz's conditions for possessing property P, and hence there may be defined over it a real continuous function assuming any value only a finite number of times. Then, as H can be transformed by a continuous function of finite sections into D, a real continuous functions of finite sections may be defined over H, and H has property P.

6. If H has property P, then the end points of H are countable, from Čech's results.

7. In general, if H has property P, then $M[X]$ need not be vacuous, where $M[X] = \Phi'_{\lambda(x)}[X] = \Phi'_{\lambda(x)+1}[X]$. For example, divide the unit interval into thirds, and on the middle third construct the upper half of a circle of radius $\frac{1}{2.3}$. Construct a normal to the circle, at its maximum point, of length $\frac{1}{2.3}$. Divide the two remaining intervals into thirds, and construct on the middle interval of each a semicircle of radius $\frac{1}{2.3^2}$, with normals of length $\frac{1}{2.3^2}$. Continue this indefinitely.

If each of the normals is bent to the right so that the end point is directly over the midpoint of the radius on the base, this space may be projected by lines perpendicular to the interval on to the interval. No more than three points of the space will be projected into any one point of the interval. This is a continuous transformation of finite sections.

Thus this space H has property P. Every nondegenerate cyclic element is a branch element, and no other element is a branch element. Moreover, every branch cyclic element is a bilateral limit element of branch elements, and $X = \Phi_1'[X] = \Phi_2'[X] = \cdots$. Thus the set of branch elements of H is not Φ'-reducible. However, H does fulfill the necessary conditions of Čech; i. e., H is a regular curve and has a countable number of end points, and the points of order greater than two form a set whose closure is totally disconnected. The set H fulfills the conditions that each cyclic element have property P, that the cut points of H on any cyclic element go into a reducible set under the continuous transformation of finite sections which may be defined on the element, but the set of branch elements is not Φ'-reducible. Thus it is seen that the collection of sets fulfilling the sufficient conditions given in paragraph 5 and hence having property P forms a proper subcollection of the collection of all sets having property P.

Let us consider now a locally connected continuum H having property P, and assume that $M[X] \neq 0$, where $M[X] = \Phi'_{\lambda(x)}[X]$. Then $M[X]$ must contain an infinite number of branch elements of H, each a bilateral limit element of $M[X]$. Let us add the restricting condition that each element Q of $M[X]$ is a trilateral limit element of $M[X]$ (definition analogous to that of bilateral limit element).

If H is transformed into an arc t by a continuous transformation $f[x]$ of finite sections, as only a finite number of elements of $M[X]$ can go into intervals of t containing an end point of t, there must be at least one interval Q_1 of $M[X]$, such that $f[Q_1] = Q_1'$ ist interior to t.

Then $H - Q_1 = T_{11}^* + T_{12}^* + T_{13}^* + \cdots$, where T_{1i}^* is a component of $H - Q_1$. At least three of the components, T_{1i}^*, T_{1j}^* and T_{12}^*, must contain elements of $M[X]$ distinct from Q_1, and as small and as close to Q_1 as we please. Then the transforms

of these components lie 1) one in Q_1', or 2) two partially exterior to Q_1' and on the same side of Q_1'. If $f[T_{1i}^*]$ lies in Q_1', designate it by T_{11}. Then $f[Q_1]$ is an arc t_1 contained in t, and $s_1 = f[T_{11}]$ is a semiopen arc. To every point of s_1 there correspond at least two points of H in Q_1 and T_{11}. If $f[T_{1i}^*]$ and $f[T_{1j}^*]$ lie partially exterior to Q_1' and on the same side, designate them by T_{11} and T_{12} respectively. Then $f[T_{11}+Q_1]\cdot f[T_{12}+Q_1]$ is an arc t_1 contained in t. To every point of $s_1 = f[T_{11}]\cdot f[T_{12}]$ there correspond at least two points of H in T_{11} and T_{12}.

Choose $Q_2 \in M[X]\cdot T_{11}$ small enough so that $f[Q_2] = Q_2'$ and $f[\bar{R}_2]$, where R_2 is a small connected region of Q_2 in T_{11}, are interior to s_1. As Q_2 is a branch element of H, it is a local branch element, and $\bar{R}_2 - Q_2 = T_{21}^* + T_{22}^* + T_{23}^* + \cdots$. At least three of the components T_{2i}^* contain elements of $M[X]$ distinct from Q_2 and as small and as close to Q_2 as we please. Then the transforms of these components lie 1) one in Q_2' or 2) two partially exterior to and on the same side of Q_2' in $f[R_2] \subset s_1$. If $f[T_{2i}^*]$ lies in Q_2', designate it by T_{21}. Then $f[Q_2]$ is an arc t_2 contained in $f[\bar{R}_2]$ and $s_2 = f[T_{21}]$ is a semiopen arc to every point of which there correspond at least three points of H, in T_{12} or Q_1, T_{21} and Q_2. If $f[T_{2i}^*]$ and $f[T_{2j}^*]$ lie partially exterior to and on the same side of Q_2', designate them by T_{21} and T_{22} respectively. Then $f[T_{21}+Q_2]\cdot f[T_{22}+Q_2]$ is an arc t_2 contained in $f[\bar{R}_2]$. To every point of $s_2 = f[T_{21}]\cdot f[T_{22}]$ there correspond at least three points of H in T_{12} or Q_1, T_{21} and T_{22}. Moreover, $\bar{s}_2 \subset f[\bar{R}_2] \subset s_1 \subset \bar{s}_1$.

In general, if we have the sets s_i so that $\bar{s}_{n-1} \subset f[\bar{R}_{n-1}] \subset \subset s_{n-2} \subset \cdots \subset \bar{s}_2 \subset f[\bar{R}_2] \subset s_1$, and $Q_1, Q_2, \ldots, Q_{n-1}$ of $M[X]$, where $Q_{n-1} \subset T_{n-2,1}$, then we choose $Q_n \in M[X]\cdot T_{n-1,1}$, small enough so that $f[Q_n] = Q_n'$ and $f[\bar{R}_n]$, where R_n is a small connected region of Q_n in $T_{n-1,1}$, are interior to s_{n-1}. As before, $\bar{R}_n - Q_n = T_{n1}^* + T_{n2}^* + T_{n3}^* + \cdots$. At least three of the components, T_{ni}^*, T_{nj}^* and T_{n12}^*, must contain elements of $M[X]$ distinct from Q_n and as small and as close to Q_n as we please. Then the transforms of these components lie 1) one interior to Q_n' or 2) two partially exterior to Q_n' and on the same side of

Q_n' in $f[R_n]$. If $f[T_{ni}^*]$ lies in Q_n', designate it by T_{n1}. Then $f[Q_n]$ is an arc t_n contained in $f[\bar{R}_n]$. To every point of $s_n = f[T_{n1}]$ there correspond at least $n+1$ points of H, from T_{12} or Q_1, T_{22} or Q_2, ... $T_{n-1,2}$ or Q_{n-1}, T_{n1} and Q_n. If $f[T_{ni}^*]$ and $f[T_{nj}^*]$ lie partially exterior to and on the same side of Q_n', designate them by T_{n1} and T_{n2}, respectively. Then $f[T_{n1}+Q_n]\cdot f[T_{n2}+Q_n]$ is an arc t_n contained in $f[\bar{R}_n]$. To every point of $s_n = f[T_{n1}]\cdot f[T_{n2}]$ there correspond at least $n+1$ points of H, in T_{12} or Q_1, ... T_{n-12} or Q_{n-1}, T_{n1} and T_{n2}. As before $\bar{s}_n \subset f[R_n] \subset s_{n-1} \subset \bar{s}_{n-1}$.

If $F = \prod_{n=1}^{\infty} s_n$, then F is not vacuous. Let $y' \in F$. Then y' is the image of a point x_n in T_{n2} or Q_n for every n. Moreover, $T_{n2}\cdot T_{m2} = T_{n2}\cdot Q_m = Q_m \cdot Q_n = 0$ for $n \neq m$. But $x_n \subset f^{-1}[y']$. Thus $f^{-1}[y]$ is not finite, contrary to hypothesis. Therefore $M[X]$ must be vacuous and H must be Φ'-reducible with respect to ist branch elements, if it has property P and satisfies the further restriction put upon the branch elements.

8. The results of paragraphs 5, 6 and 7 may be combined in the following theorems:

Theorem 1. *If every cyclic element C of a locally connected continuum H has the property that over it may be defined a real continuous function assuming any value only a finite number of times, such that the set of cut points of H on C is transformed into a reducible set, in order that such a function may be defined over the whole set H, it is sufficient that*

a) the end points of H be countable, and

b) the set of branch elements of H be reducible by the operation Φ' of bilateral coherence, defined in paragraph 5.

Theorem 2. *If the cyclic elements of H satisfy the conditions of Theorem 1, and if, moreover, when Φ' is the operation of bilateral coherence, all elements of $\Phi'_{\lambda(x)}[X] = \Phi'_{\lambda(x)+1}[X]$, where X is the set of branch elements of H, are trilateral limit elements of $\Phi'_{\lambda(x)}[X]$, in order that such a function may be defined over the whole set H, it is necessary and sufficient that*

a) the end points of H be countable, and

b) the set of branch elements of H be reducible by the operation of bilateral coherence.

9. If we could find suitable conditions under which a cyclicly connected space could be transformed into a simple arc by a continuous transformation of finite sections, we should be able to state much more concerning conditions to be imposed on Peano spaces in order that they have property *P*.

A very restricting condition of this type is that the cyclicly connected space be locally a dendrite; i. e., for any point p of the space and some $\varepsilon > 0$, the closure of some ε-region of p is a dendrite. If this is the case at every point of the space, the space contains but a finite number of points of order greater than two. If there were an infinite number of such points, a sequence of them would have a limit point at which the space would be either not locally connected or not locally a dendrite, both possibilities not allowed.

It is easily proved that if all the nondegenerate cyclic elements of a space are locally dendrites, there is definable over each of them a continuous function of finite sections. If they have the further property that the set of cut points on each goes into a reducible set under some such continuous transformation, then if the space fulfills the conditions of theorem 1, the whole space will have property *P*.

However, this case is particularly simple. There is a long step between it and the next simplest case, where the cyclic element is locally a dendrite at all but a finite or countable number of points.

If the whole problem could be solved for cyclicly connected spaces, and if the conditions of reducibility of the cut points on a cyclic element and of Φ'-reducibility of the branch elements could be weakened to come thing both necessary and sufficient, the problem of Čech would be completely solved.

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